Supersolid in Bose-Bose-Fermi Mixtures subjected to a Square Lattice

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Two-component Bose condensates with repulsive interaction are stable when $g_1 g_2 < g_{12}^2$ is satisfied. By tuning the interactions, we show that the instability corresponding to bose-bose phase separation always happens at a higher temperature than corresponding to bose-fermi phase separation happens. Moreover, we find both the transition temperature $T_{\rm DW}$ of supersolid and the coherence peak at $k_{\rm DW}$ are enhanced in the mixtures studied. These will make the observation of supersolid in experiments more reachable.

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1. INTRODUCTION

Supersolids, a concept simultaneously exhibiting superfluidity and crystalline order, have been studied intensely over five decades [1–4]. Theoretically, people mainly focus on lattice models of interacting bosons and fermions such as the Hubbard model and its various generalizations and have obtained many important results by numerical analysis [5, 6]. Experimentally, Kim and Chan recently reported they found nonclassical rotational inertia which should be an direct evidence of supersolid based on Leggett's suggestion in solid ^{4}He [7, 8], however, it has also been pointed out that this observation may not be due to supersolid but due to other reasons, such as an increase in shear modulus of bulk solid helium [9], and triggered an intense debate [10, 11].

Besides the study of supersolids in condensed matter systems, ultracold atoms in optical lattices [12] have emerged as a parallel platform with highly controllability to study supersolids. Trapped Bose-Einstein condensates with dipole-dipole interaction can produce a "roton" minimum in the excitation spectrum [13–15], and this led to the prediction of supersolid upon softening of the roton excitation energy [16, 17]. Recently, on the basis of off-resonant dressing of atomic Bose-Einstein condensates to high-lying Rydberg states, people have found the effective atomic interactions resulting from such a Rydberg dressing can also produce a roton minimum and, therefore, provide a clean realization of available model for supersolidity [18, 19].

In this work, we consider the two kind of bosons are two hyperfine state of ^{87}Rb, and the fermions are a hyperfine state of ^{40}K and investigate bose-bose-fermi mixtures in a square lattice. For the bose-fermi mixtures subjected to a square lattice, it has been pointed out that the density wave instability introduced by fermions will establish crystalline order, while the condensate bosons exhibit superfluidity, so a supersolid phase emerges at finite temperature [17]. For the bose-bose-fermi mixtures studied here, besides the density wave instability introduced by fermions, there is another instability between the two-component bose-condensates when $g_1 g_2 = g_{12}^2$, where $g_{1,2}$ are the repulsive intraspecies interaction and g_{12} is the interspecies interaction [20]. When $g_1 g_2 > g_{12}^2$, the bose-condensates are mixed and stable. When $g_1 g_2 < g_{12}^2$, the bose-condensates are unstable and tend to either phase separation or collapse depending on $g_{12} > 0$ or < 0. In this article, we assume the bose-condensates are initially mixed and stable, and we find that bose-bose phase separation always happens before bose-fermi phase separation when we decrease the temperature. Moreover, we find both the transition temperature $T_{\rm DW}$ of supersolid and the coherence peak at $k_{\rm DW}$ are enhanced comparing to the bose-fermi mixtures case [17].

The article is organized as follows. In Sec.2, we consider the two kind of bosons are two hyperfine state of ^{87}Rb, and the fermions are a hyperfine state of ^{40}K and investigate bose-bose-fermi mixtures in a square lattice, and give the fermionic response in the static limit. In Sec.3, we give the details of the instabilities and different phases induced by the instabilities, and give a mean field description of the supersolid phase. Some conclusions are obtained in Sec.4.

2. THE BOSE-BOSE-FERMI MIXTURES IN A SQUARE LATTICE

The Hamiltonian for the bose-bose-fermi mixtures takes the form $H = H_0 + H_{int}$ with ($\alpha =\uparrow, 0, \downarrow$)

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$$H_0 = \sum_\alpha \int d\mathbf{x}\psi^\dagger_\alpha \left\{(-\frac{\hbar^2}{2m_\alpha}\bigtriangledown^2 + V_\alpha(x)\right\}\psi_\alpha,$$
$$H_{int} = \int d\mathbf{x}\left\{g_1\psi^\dagger_\uparrow\psi^\dagger_\uparrow\psi_\uparrow\psi_\uparrow + g_2\psi^\dagger_\downarrow\psi^\dagger_\downarrow\psi_\downarrow\psi_\downarrow \right. \quad (1)$$
$$\left. +2g_{12}\psi^\dagger_\uparrow\psi^\dagger_\downarrow\psi_\downarrow\psi_\uparrow + 2g_{\rm BF}\left(\psi^\dagger_\uparrow\psi_\uparrow + \psi^\dagger_\downarrow\psi_\downarrow\right)\psi^\dagger_0\psi_0\right\},$$

where $\psi^\dagger_{\uparrow,\downarrow}$ are the bosonic field operators and $\psi^\dagger_0$ is the fermionic field operator. In order to assure the mixtures to be stable, we assume all of the interactions between bosons are repulsive, with $g_{\alpha\beta} = 4\pi a_{s,\alpha\beta}\hbar^2/m$ ($\alpha,\beta = 1,2$. In this work, we use α,β to label the interactions, densities and phases of bosons, and use $\uparrow,\downarrow$ only to label the bosonic operators, moreover, when $\alpha = \beta$ we only keep α for convenience) and $g_1g_2 > g^2_{12}$. $g_{\rm BF} = 2\pi a_{\rm BF}\hbar^2/\mu$, the strength of coupling between bosons and fermions, we assume that they are equal for both component of bosons. $a_{s,\alpha}$ is the intraspecies scattering length, $a_{s,12}$ is the interspecies scattering length, and μ is the relative mass. $V_\alpha(\mathbf{x}) = V_\alpha[\sin^2(\pi x/a) + \sin^2(\pi y/a)]$ is the periodic potential produced by the optical lattice with wave-length $\lambda = 2a$ and m_α are the mass of bosons and fermions. As the bosons are two hyperfine state of ^{87}Rb, it is justified to assume $m_\uparrow = m_\downarrow$ and $V_\uparrow = V_\downarrow$ for simplicity in the following. Since the fermions are single component, the interaction between them can be neglected due to Pauli exclusion principle.

In order to obtain the Hamiltonian in momentum space, we follow the procedures used in Ref.[17] and expand the bosonic and fermionic field operators ψ_α in the forms

$$\psi_{\uparrow,\downarrow}(x) = \sum_{\mathbf{k}\in K} b_{\mathbf{k}\uparrow,\downarrow}w_{\mathbf{k}\uparrow,\downarrow}(x),$$
$$\psi_0(x) = \sum_{\mathbf{k}\in K} c_{\mathbf{k}}v_{\mathbf{k}}(x), \quad (2)$$

where K denotes the first Brillouin zone, $b_{\mathbf{k}\uparrow,\downarrow}$ and $c_{\mathbf{k}}$ are the bosonic and fermionic annihilation operators, while $w_{\mathbf{k}\uparrow,\downarrow}(x)$ and $v_{\mathbf{k}}(x)$ are the Bloch wave functions corresponding to a single boson ($\uparrow$ or $\downarrow$) or fermion in the periodic potential V_α, respectively. Since $m_\uparrow = m_\downarrow$ and $V_\uparrow = V_\downarrow$, $w_{\mathbf{k}\uparrow}(x)$ should be equal to $w_{\mathbf{k}\downarrow}(x)$. Therefore, we use $w_{\mathbf{k}}(x)$ to denote both of them for convenience. Substituting Eq.(2) into Eq.(1) and restricting in the lowest Bloch band, we obtain the Hamiltonian in momentum space as

$$H = \sum_{\mathbf{k}\in K,\sigma}\epsilon_{{\rm B}\sigma}b^\dagger_{\mathbf{k}\sigma}b_{\mathbf{k}\sigma} + \sum_{\{\mathbf{k},\mathbf{k}',\mathbf{q},\mathbf{q}',\sigma\}}\frac{U_{{\rm B}\sigma}}{2N}b^+_{\mathbf{k}\sigma}b_{\mathbf{k}'\sigma}b^+_{\mathbf{q}\sigma}b_{\mathbf{q}'\sigma}$$
$$+\sum_{\{\mathbf{k},\mathbf{k}',\mathbf{q},\mathbf{q}'\}}\frac{U_{\rm B,12}}{N}b^+_{\mathbf{k}\uparrow}b_{\mathbf{k}'\uparrow}b^+_{\mathbf{q}\downarrow}b_{\mathbf{q}'\downarrow} + \sum_{\mathbf{q}\in K}\epsilon_{\rm F}(\mathbf{q})c^+_{\mathbf{q}}c_{\mathbf{q}}$$
$$+\frac{U_{\rm BF}}{N}\sum_{\{\mathbf{k},\mathbf{k}',\mathbf{q},\mathbf{q}',\sigma\}}b^+_{\mathbf{k}\sigma}b_{\mathbf{k}'\sigma}c^+_{\mathbf{q}}c_{\mathbf{q}'}, \quad (3)$$

where N is the number of unit cells, $\epsilon_{\rm F,B\sigma}(\mathbf{k})$ denote the energy dispersion of the fermions and bosons, respectively, while $U_{\rm BF} = g_{\rm BF}\int d\mathbf{x}|\widetilde{w}|^2|\widetilde{v}|^2$ and $U_{{\rm B},\alpha\beta} = g_{\alpha\beta}\int d\mathbf{x}|\widetilde{w}|^4$, with $\widetilde{w}(\mathbf{x})$ and $\widetilde{v}(\mathbf{x})$, the Wannier functions associated with the Bloch band $w_{\mathbf{k}}(\mathbf{x})$ and $v_{\mathbf{k}}(\mathbf{x})$. In a deep optical lattice, the Wannier functions $\widetilde{w}(\mathbf{x})$ and $\widetilde{v}(\mathbf{x})$ are well localized around the minimum of V_α. As a result, the Hamiltonian reduces to a familiar Bose-Fermi-Hubbard model, and for $\epsilon_{\rm F,B\sigma}(\mathbf{k})$, only nearest neighbor hopping survives,

$$\epsilon_{\rm B}(\mathbf{q}) = 2J_{\rm B}\left[2 - \cos(q_xa) - \cos(q_ya)\right],$$
$$\epsilon_{\rm F}(\mathbf{q}) = -2J_{\rm F}\left[\cos(q_xa) + \cos(q_ya)\right], \quad (4)$$

where $J_{\rm B,F}$ is the hopping energy for fermions and bosons, respectively. The bosonic dispersion relation implies $\mu_B = -4J_B$ and the bosons will form a zero-momentum Bose-Einstein condensation for sufficiently low temperature. The fermionic dispersion relation implies the Fermi surface at half-filling $n_{\rm F} = 1/2$ (where $\mu_{\rm F} = 0$, in this work. $n_{\rm F}$ and $n_{\rm B}$ denote the number of particles per unit cell) and exhibits perfect nesting for $\mathbf{k}_{\rm DW} = (\pi/a,\pi/a)$ and van Hove singularities at $\mathbf{k} = (0,\pm\pi/a),(\pm\pi/a,0)$.

Integrating out the fermions produces two effects. To the first order in $U_{\rm BF}$ (in this work, we focus on weak interaction, so an expansion in $U_{\rm BF}$ and a cut at the second order are justified), the fermions simply produce a (trivial) shift of the bosonic chemical potential $\mu_{{\rm B}\sigma} \to \mu_{{\rm B}\sigma} - U_{\rm BF}n_{\rm F}$. To the second order in $U_{\rm BF}$, the fermions provide an effective interaction for the bosons which depends on the temperature T of the fermionic atom gas,

$$H_{\rm int} = \frac{1}{2N}\sum_{\{\mathbf{k},\mathbf{k}',\mathbf{q},\mathbf{q}',\sigma\}}U_{{\rm B}\sigma}(T,\mathbf{q}-\mathbf{q}')b^+_{\mathbf{k}\sigma}b_{\mathbf{k}'\sigma}b^+_{\mathbf{q}\sigma}b_{\mathbf{q}'\sigma}$$
$$+\frac{1}{N}\sum_{\{\mathbf{k},\mathbf{k}',\mathbf{q},\mathbf{q}',\}}U_{\rm B,12}(T,\mathbf{q}-\mathbf{q}')b^+_{\mathbf{k}\uparrow}b_{\mathbf{k}'\uparrow}b^+_{\mathbf{q}\downarrow}b_{\mathbf{q}'\downarrow},$$

with

$$U_{{\rm B}\sigma}(T,\mathbf{q}-\mathbf{q}') = U_{{\rm B}\sigma} + U^2_{\rm FB}\chi(T,\mathbf{q}-\mathbf{q}'),$$
$$U_{\rm B,12}(T,\mathbf{q}-\mathbf{q}') = U_{\rm B,12} + U^2_{\rm FB}\chi(T,\mathbf{q}-\mathbf{q}'). \quad (5)$$

The fermionic response in the static limit is given by the Lindhard function

$$\chi(T,\mathbf{q}) = \int_K \frac{d\mathbf{k}}{v_0}\frac{f[\epsilon_{\rm F}(\mathbf{k})] - f[\epsilon_{\rm F}(\mathbf{k}+\mathbf{q})]}{\epsilon_{\rm F}(\mathbf{k}) - \epsilon_{\rm F}(\mathbf{k}+\mathbf{q}) + i\eta}, \quad (6)$$

where $v_0 = (2\pi/a)^2$ is the volume of the first Brillouin zone, $f(\epsilon) = 1/[1 + \exp(\epsilon/T)]$ ($\mu_{\rm F} = 0$ at half filling) is just the Dirac-Fermi distribution function. The static limit is justified if the fermions are much faster than the bosons ($J_{\rm F} >> J_{\rm B}$), as then the fermionic response occurs on much faster timescales than the movement of the bosons, and one can safely neglect retardation effects

[21]. Using the fermionic dispersion relation Eq.(4), the Lindhard function exhibits two logarithmic singularities at $\mathbf{q} = 0$ and $\mathbf{k}_{\rm DW}$. The singularity at $\mathbf{q} = 0$ is purely due to the logarithmic van Hove singularity in the density of states, and the singularity at $\mathbf{k}_{\rm DW}$ is due to the combination of van Hove singularities and perfect nesting. The singularity at $\mathbf{q} = 0$ induces an instability towards a series of phase separation, while the singularity at $\mathbf{k}_{\rm DW}$ induces an instability towards density wave formation and provides a supersolid phase. The two instabilities are competing with each other.

3. INSTABILITIES AND PHASES

For the weak interaction, when the temperatures is well below the superfluid transition temperature $T_{\rm KT}$ of the bosons, the Lindhard function at $\mathbf{q} = 0$ reduces to $\chi(T \to 0, 0)$ and takes the form [17]

$$\chi(T \to 0, 0) = \int d\epsilon N(\epsilon)\partial_\epsilon f(\epsilon) \sim -N_0 \ln \frac{16 c_1 J_{\rm F}}{T}, \quad (7)$$

with $N(\epsilon) \sim N_0 \ln |16 J_{\rm F}/\epsilon|$, $N_0 = 1/(2\pi^2 J_{\rm F})$ and $c_1 = 2\exp(C)/\pi \approx 1.13$. As $\chi(T \to 0, 0)$ is always negative, the coupling between the bosons and the fermions induces an attractive interaction, which is proportional to $U_{\rm FB}^2 \chi(T, \mathbf{0})$, between the bosons (see Eq.(5)). This attractive interaction has the effect to reduce the repulsive interactions $U_{B,\alpha\beta}$ between bosons to $U_{{\rm eff},\alpha\beta} = U_{{\rm B},\alpha\beta} + U_{\rm FB}^2 \chi(T, 0)$. As a result, even $U_{\rm B,1} U_{\rm B,2} > U_{\rm B,12}^2$ (equivalent to $g_1 g_2 > g_{12}^2$) initially, $U_{\rm eff,1} U_{\rm eff,2}$ can be tuned to equal to $U_{\rm eff,12}^2$ by lowering the temperature to some value. Moreover, a superfluid condensate at low temperatures to be stable requires a positive effective interaction $U_{{\rm eff},\alpha} > 0$. If we take $U_{\rm B,1}$ as the energy unit, and define the ratios $U_{\rm B,2}/U_{\rm B,1}$, $U_{\rm B,12}/U_{\rm B,1}$ and $U_{\rm BF}/U_{\rm B,1}$ as γ, λ and κ, respectively. The condition $U_{eff,12}^2 = U_{eff,1} U_{eff,2}$ defines the critical temperature $T_{\rm BB,PS}$ for bose-bose phase separation,

$$T_{\rm BB,PS} = 16 c_1 J_{\rm F} \exp\left[\frac{\lambda^2 - \gamma}{N_0 \kappa^2 (1 + \gamma - 2\lambda)}\right]. \quad (8)$$

The condition $U_{{\rm eff},\alpha} = 0$ defines two critical temperatures $T_{\rm BF1,PS}$ and $T_{\rm BF2,PS}$ for bose-fermi phase separation.

$$\begin{aligned} T_{\rm BF1,PS} &= 16 c_1 J_{\rm F} \exp\left[-\frac{1}{N_0 \kappa^2}\right], \\ T_{\rm BF2,PS} &= 16 c_1 J_{\rm F} \exp\left[-\frac{\gamma}{N_0 \kappa^2}\right]. \end{aligned} \quad (9)$$

When $\lambda \neq 1$ and γ, it is directly to show that $(\gamma - \lambda^2)/(1 + \gamma - 2\lambda)$ is always smaller than $\min\{1, \gamma\}$ under the constraint $\lambda^2 < \gamma$, which is the condition that the bose condensates are initially mixed (see Fig.1). $(\gamma - \lambda^2)/(1 + \gamma - 2\lambda) < \min\{1, \gamma\}$ indicates when we lower the temperature, the bose-bose mixtures are always easier to be unstable and phase separated (or collapse, see Fig.1) than the bose-fermi mixtures. Moreover, when λ gets close to the boundary $\sqrt{\gamma}$, $(\gamma - \lambda^2)/(1 + \gamma - 2\lambda)$ decreases very fast, as a result, $T_{\rm BB,PS}$ increases exponentially to values much larger than $max\{T_{\rm BF1,PS}, T_{\rm BF2,PS}\}$ and easy to reach in experiments. Therefore, such a bose-bose phase separation induced by fermions should be easy to be observed in experiments. If we continue to lower the temperature after the bose-bose mixtures are phase separated, we can expect that bose-fermi phase separation will happen and all the components will distribute separately in space at last.

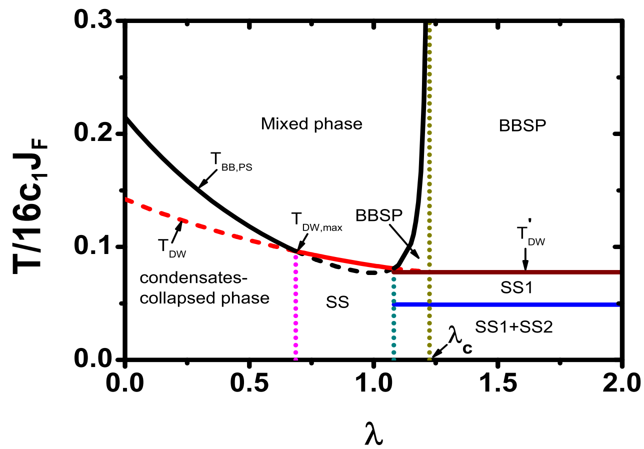

FIG. 1: (Color online) $\lambda - T$ Phase diagram. Parameters are set as $t_{\rm B} = 0.55$, $N_0 \kappa^2 = 0.39$, $\gamma = 1.5$, $\lambda_c = \sqrt{\gamma}$. For $\lambda > \lambda_c$, the two-component bosons are phase separation initially (BBPS), and supersolid corresponding to bosons-↑ (SS1) emerges when the temperature is below T'_{DW}. For $\lambda < \lambda_c$, the supersolid phase established by the two Bose-condensates appears when the temperature is below $T_{\rm DW}$ with $T_{\rm BB,PS} < T_{\rm DW}$. For $\lambda < 1$ ($1 < \lambda < \lambda_c$), the two Bose-condensates collapse (separate) when the temperature is below $T_{\rm BB,PS}$ with $T_{\rm BB,PS} > T_{\rm DW}$.

Now, we discuss the second instability induced by the singularity in the Lindhard function at $\mathbf{k}_{\rm DW}$. Using Eq.(6) and the perfect nesting $\epsilon_{\rm F}(\mathbf{q} + \mathbf{k}_{\rm DW}) = -\epsilon_{\rm F}(\mathbf{q})$, the Lindhard function becomes [17]

$$\chi(T, \mathbf{k}_{\rm DW}) = \int d\epsilon N(\epsilon) \frac{\tanh(\epsilon/2T)}{-2\epsilon} \sim -\frac{N_0}{2}\left[\ln \frac{16 c_1 J_{\rm F}}{T}\right]^2. \quad (10)$$

The combination of van Hove singularities and perfect nesting produces a $[\ln T]^2$ singular behavior. Such a singular behavior can produce a roton minimum at $\mathbf{k}_{\rm DW}$. Within Bogoliubov theory, the bosonic quasi-particle

spectrum becomes

$$E^2_{\mathrm{B},\pm}(\mathbf{q}) = \epsilon^2_{\mathrm{B}}(\mathbf{q}) + \epsilon_{\mathrm{B}}(\mathbf{q})n_{\mathrm{B}}\left[U_{\mathrm{B1}}(T,q)+U_{\mathrm{B2}}(T,q)\right] \pm \Big\{\epsilon^2_{\mathrm{B}}(\mathbf{q})n^2_{\mathrm{B}}\left[U_{\mathrm{B1}}(T,q)-U_{\mathrm{B2}}(T,q)\right]^2 + 4\epsilon^2_{\mathrm{B}}(\mathbf{q})n^2_{\mathrm{B}}U^2_{\mathrm{B},12}(T,q)\Big\}^{1/2}, \quad (11)$$

here we have assumed $n_{\mathrm{B1}} = n_{\mathrm{B2}} = n_{\mathrm{B}}$. The induced attraction proportional to $U^2_{\mathrm{BF}}\chi(T,\mathbf{k}_{\mathrm{DW}})$ reduces the energy of quasi-particles at $\mathbf{k}_{\mathrm{DW}}$ from a pure-bosonic maximum (when $U_{\mathrm{BF}} = 0$, the maximum of $E_{\mathrm{B},-}(\mathbf{q})$ locates at $\mathbf{k}_{\mathrm{DW}}$) to an induced zero roton minimum ($E_{\mathrm{B},-}(\mathbf{k}_{\mathrm{DW}})) = 0$) at the critical temperature

$$T_{\mathrm{DW}} = 16c_1 J_{\mathrm{F}} \exp\left[-\sqrt{\frac{t^2_{\mathrm{B}} + 2t_{\mathrm{B}}(1+\gamma) + 4\gamma - 4\lambda^2}{2N_0\kappa^2(1+\gamma+t_{\mathrm{B}}-2\lambda)}}\right] \quad (12)$$

with $t_{\mathrm{B}} = 8J_{\mathrm{B}}/n_{\mathrm{B}}U_{\mathrm{B1}}$. As $E_{\mathrm{B},-}(\mathbf{k}_{\mathrm{DW}}, T_{\mathrm{DW}}) = E_{\mathrm{B},-}(\mathbf{k}=\mathbf{0}) = 0$, we can expect the boson modes $b_{\mathbf{k}_{\mathrm{DW}}\alpha}$ to become macroscopically occupied just like the boson mode $b_{0\alpha}$ below this critical temperature. Comparing this result to the one obtained in Ref.[17],

$$T'_{\mathrm{DW}} = 16c_1 J_{\mathrm{F}} \exp\left[-\sqrt{(2+t_{\mathrm{B}})/\lambda_{\mathrm{FB}}}\right],$$

we find T_{DW} is always higher than T'_{DW} when parameters appearing in both systems take the same values (see Fig.1). Moreover, since T_{DW} depends on $\sqrt{\frac{t^2_{\mathrm{B}}+2t_{\mathrm{B}}(1+\gamma)+4\gamma-4\lambda^2}{2N_0\kappa^2(1+\gamma+t_{\mathrm{B}}-2\lambda)}}$ exponentially (12), a small change of this term may induce a great change of T_{DW}. Therefore, such an enhancement of critical temperature can be large. However, T_{DW} can not increase as greatly as $T_{\mathrm{BB,PS}}$, since t_{B} has to be larger than a critical value $t_{\mathrm{SF-MI}} \approx 1/3$, below which Mott insulating phase emerges and the above picture fails [12]. As a comparison, we calculate T_{DW} based on the parameters used in Ref.[17] and find $T_{\mathrm{DW,max}}/T'_{\mathrm{DW}} \approx 1.3$ (this ratio goes to the maximum when $\gamma \to 1$, in Fig.1, we take $\gamma = 1.5$ just for manifesting every phase) under the constraint $T_{\mathrm{DW,max}} > T_{\mathrm{BB,PS}}$ (if bose-bose phase separation happens first, T_{DW} reduces to T'_{DW}, and the enhancement effect of T_{DW} misses). Such an enhancement of T_{DW} is of realistic meaning, since the lower the temperature is, the harder it is to reach in cold atomic experiments.

For temperatures well below T_{KT} and T_{DW}, both the boson mode $b_{\mathbf{k}_{\mathrm{DW}}\alpha}$ and $b_{0\alpha}$ are macroscopically occupied, therefore, it is justified to use mean fields $\langle b_{\mathbf{k}_{\mathrm{DW}}\alpha}\rangle$ and $\langle b_{0\alpha}\rangle$ to substitute them. Introducing the mean fields $\langle b_{0\alpha}\rangle = \sqrt{n_{0\alpha}N}\exp(i\varphi_{0\alpha})$ and $\langle b_{\mathbf{k}_{\mathrm{DW}}\alpha}\rangle = (\Delta_\alpha/2U_{\mathrm{BF}})\sqrt{N/n_{0\alpha}}\exp(i\varphi_\alpha)$ with the constraint $n_{\mathrm{B}\alpha} = n_{0\alpha} + \Delta^2_\alpha/(4n_{0\alpha}U^2_{\mathrm{BF}})$ and neglecting thermal excitations of bosonic quasi-particles [17], we obtain the bosonic densities as

$$n_{\mathrm{B}\alpha}(x,y) = n_{\mathrm{B}\alpha} + \frac{\Delta_\alpha\cos\theta_\alpha}{U_{\mathrm{BF}}}\left[\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}\right] \quad (13)$$

with $\theta_\alpha = \varphi_{0\alpha} - \varphi_\alpha$. The phase difference $\Delta\theta = \theta_1 - \theta_2$ between the two bosonic density waves determines whether they are constructive or destructive. Introducing $\langle b_{0\alpha}\rangle$ and $\langle b_{\mathbf{k}_{\mathrm{DW}}\alpha}\rangle$ to Hamiltonian (3) and neglecting terms independent of Δ_α, the Hamiltonian per unit cell is given as

$$\frac{H}{N} = 2J_{\mathrm{B}}\frac{\Delta_1^2+\Delta_2^2}{n_{\mathrm{B}}U^2_{\mathrm{BF}}} + \frac{U_{\mathrm{B1}}\Delta_1^2\cos^2\theta_1}{2U^2_{\mathrm{BF}}} + \frac{U_{\mathrm{B2}}\Delta_2^2\cos^2\theta_2}{2U^2_{\mathrm{BF}}} + \frac{U_{\mathrm{B},12}\Delta_1\Delta_2(\cos^2\theta_1+\cos^2\theta_2+2\cos(\theta_1-\theta_2))}{4U^2_{\mathrm{BF}}} + \frac{H_{\mathrm{F}}}{N} + o(\Delta^4). \quad (14)$$

The terms in the first and second lines describe the increase in the kinetic and interaction energies of the bosons due to the modulation of densities triggered by the boson modes $b_{\mathbf{k}_{\mathrm{DW}}\alpha}$, while H_{F} takes the form

$$H_{\mathrm{F}} = \frac{1}{2}\sum_{\mathbf{q}\in K}\left(\, c^+_{\mathbf{q}},\ c^+_{\mathbf{q}'}\,\right)\begin{pmatrix}\epsilon_{\mathrm{F}}(\mathbf{q}) & \Delta(\theta_1,\theta_2)\\ \Delta(\theta_1,\theta_2) & \epsilon_{\mathrm{F}}(\mathbf{q}')\end{pmatrix}\begin{pmatrix}c_{\mathbf{q}}\\ c_{\mathbf{q}'}\end{pmatrix} \quad (15)$$

with a constraint $\mathbf{q}' = \mathbf{q} - \mathbf{k}_{\mathrm{DW}} + \mathbf{K}_h$ (the reciprocal lattice vector $\mathbf{K}_h$ ensures the constraint $\mathbf{q}' \in K$) and $\Delta(\theta_1,\theta_2) = \Delta_1\cos\theta_1 + \Delta_2\cos\theta_2$. Diagonalizing the fermionic Hamiltonian, we obtain the fermionic quasi-particle excitation spectrum $E_{\mathrm{F}}(\mathbf{k},\Delta) = \pm[\epsilon^2_{\mathrm{F}}(\mathbf{k}) + (\Delta_1\cos\theta_1 + \Delta_2\cos\theta_2)^2]^{1/2}$. To determine the phase difference $\Delta\theta$, we minimize the thermodynamic potential $\Omega(T,\Delta_1,\Delta_2,\theta_1,\theta_2)$ and find a constraint between θ_1 and θ_2: $\theta_1 = \theta_2 = s\pi$, with s an integer. Therefore, the phase difference $\Delta\theta = 0$, the two bosonic density waves are completely constructive and produce a stronger density wave. A stronger density wave makes the crystalline order favorable, therefore, such a phase-locking effect is favorable to form a supersolid phase. As $\theta_1 = \theta_2 = s\pi$, $\Delta_+ = \Delta_1 + \Delta_2$ is in fact the gap. Introducing $\Delta_\pm = \Delta_1 \pm \Delta_2$ and rewriting Eq.(14), the self-consistency relations ($\partial_{\Delta_\pm}\Omega = 0$) take the form

$$\begin{aligned}&(1+\gamma+t_B-2\lambda)\Delta_- = (\gamma-1)\Delta_+,\\ &\frac{1}{2N_0\kappa^2}[t_B + (1+\frac{\Delta_-}{\Delta_+}) + \gamma(1-\frac{\Delta_-}{\Delta_+}) + 2\lambda] =\\ &\frac{1}{N_0}\int_K\frac{d\mathbf{k}}{v_0}\frac{\tanh\left[E_{\mathrm{F}}(\mathbf{k},\Delta_+)/2T\right]}{E_{\mathrm{F}}(\mathbf{k},\Delta_+)}.\end{aligned} \quad (16)$$

Setting $\Delta_+(T_{\mathrm{DW}}) = 0$ and combining the two equations above, we reproduce the critical temperature in Eq.(12). This confirms the picture that upon softening $E_{\mathrm{B},-}(\mathbf{k}_{\mathrm{DW}})$ to zero the bosonic density waves characterized by $\langle b_{\mathbf{k}_{\mathrm{DW}}\alpha}\rangle \neq 0$ emerge with a breaking of the discrete symmetry of the optical lattice be right. Furthermore, using the density of states $N_{\Delta_+}(\epsilon) =$

$N(\sqrt{\epsilon^2 - \Delta_+^2})|\epsilon|/\sqrt{\epsilon^2 + \Delta_+^2}$, the gap at $T = 0$ becomes

$$\Delta_+(0) = 32J_{\text{F}} \exp\left[-\sqrt{\frac{t_{\text{B}}^2 + 2t_{\text{B}}(1+\gamma) + 4\gamma - 4\lambda^2}{2N_0\kappa^2(1+\gamma+t_{\text{B}}-2\lambda)}}\right], \tag{17}$$

and the standard BCS relation $2\Delta_+(0)/T_{\text{DW}} = 2\pi/e^C \approx 3.58$ holds. This relation implies that the density wave have the characteristic of the superfluid, an evidence of supersolid. Therefore, T_{DW} is just the critical temperature of supersolid to emerge.

In experiments, the supersolid can be detected via the usual coherence peak of a bosonic condensate in an optical lattice. The appearance of a coherence peak at $\mathbf{k}_{\text{DW}}$ is a symbol that the supersolid appears. Since the weight of this coherence peak is proportional to the number of bosons condensed at $\mathbf{k}_{\text{DW}}$, the larger Δ_+ (here equivalent to $\langle b_{\mathbf{k}_{\text{DW}}\alpha}\rangle$) is, the sharper the peak is. Therefore, based on the similarity of the forms between $\Delta_+(0)$ and T_{DW}, we find a sharper coherence peak at $\mathbf{k}_{\text{DW}}$ appears in bose-bose-fermi mixtures compared to the one appearing in bose-fermi mixtures [17] when parameters appearing in both systems take the same values. Based on the results above, we can make the conclusion that it is more favorable to observe the supersolid in bose-bose-fermi mixtures than in bose-fermi mixtures.

4. CONCLUSIONS

In this article, we have investigated a bose-bose-fermi mixture subjected to a square lattice and found that the instability corresponding to bose-bose phase separation always happens at a higher temperature than the one corresponding to bose-fermi phase separation. Moreover, we find both the transition temperature T_{DW} of supersolid and the coherence peak at k_{DW} are enhanced in the mixtures studied. These will make the observation of supersolid in experiments more reachable.

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